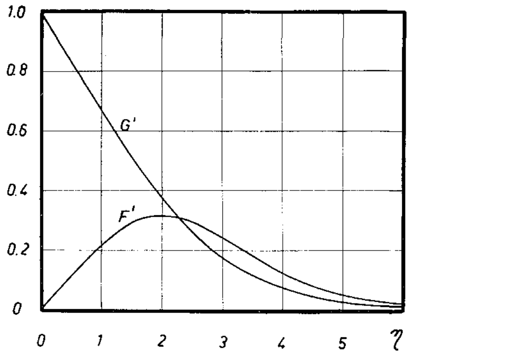

Fig. 2 Velocity profiles for plane-wall jet over a sliding plate.

nificance. [A similar situation arises for the two-dimensional eigenvalue problem (Glauert problem): however, in this case the reverse-flow solutions mentioned by Glauert have not been determined yet].

Consider now the case of a plane jet flowing (in the x direction) over both sides of a semi-infinite plate that moves with a constant velocity (in the y direction). In this case the pertinent boundary conditions are $v(0) = V$; $v(\infty) = 0$ so that, as Eq. (1) shows, $m = 0$ and $G'(0) = H(0) = 1$; $G(\infty)' = H(1) = 0$. The corresponding solutions for G' and G are

$$G' = 1 - h \qquad (3)^{1/2}G = 6\ tn^{-1}[(1 + 2h)/(3)^{1/2}] - \pi$$

The function $G'(\eta) = v/V$ and G are shown in Figs. 2 and 3 where, for comparison, the functions F' and F are also reported. These figures are self explanatory.

The characteristics of the dissipative flow are given by

$$u = (5M/2\nu x)^{1/2}F'(\eta) \qquad v = VG'(\eta) = V(1 - h)$$

$$\eta = \left(\frac{5M}{32\nu^3x^3}\right)^{1/4} z = \ln\left[\frac{(1 + h + h^2)^{1/2}}{1 - h}\right] + (3)^{1/2}\ tn^{-1}\left[\frac{(3)^{1/2}h}{2 + h}\right]$$

$$\frac{\tau_{xz}}{\rho} = \frac{1}{9}\left(\frac{125M^3}{8\nu x^5}\right)^{1/4} \qquad \frac{\tau_{yz}}{\rho} = -\frac{V}{3}\left(\frac{5M\nu}{32x^3}\right)^{1/4}$$

$$w(\infty) = -\frac{1}{4}\left(\frac{40M\nu}{x^3}\right)^{1/4}$$

where ρ is the density, ν the kinematic viscosity, τ_{xz} and τ_{yz} the shear stresses at the wall, and M is a constant related to the "exterior moment flux"[2] of the jet. The expressions for u, η, and τ_{xz} are those given by Glauert.[2] The motion of the plate does not have any influence on the flux of mass ρw entrained into the dissipative region.

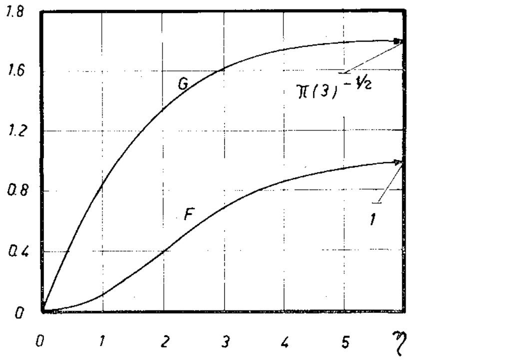

Fig. 3 Functions G and F for plane-wall jet over a sliding plate.

The equation for the limiting streamlines is given by

$$y - y_0 = \frac{2}{3} V \left(\frac{2\nu}{5M}\right)^{1/2}\left[\frac{G'(0) - 1}{F'(0)}\right] x^{3/2} = -V\left(\frac{2\nu}{5M}\right)^{1/2} x^{3/2}$$

with respect to a reference frame fixed on the plate, the "absolute" limiting streamlines being, obviously, the lines $y =$ const.

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Similar Flow Boundary Layer on Bodies in the Presence of Shear Flow

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INCOMPRESSIBLE laminar flow over two-dimensional symmetric bodies described by $y_b = \pm x_b{}^m$ is considered. y and x are dimensionless coordinates (b refers to the body). The dimensionless velocity far upstream ($x = -\infty$) is given by

$$\partial\psi/\partial y = 1 + \Omega\,|y| \qquad m < 1 \tag{1}$$

Equation (1) describes a symmetric shear flow where Ω is a dimensionless vorticity $\Omega = \omega_0 a/U_0$, ψ is a dimensionless outer-stream function, U_0 is the reference velocity, a is a reference length, and ω_0 is the uniform vorticity. The velocity component tangent to the body surface is given by the superposition of two contributions. Uniform flow contributes a velocity that is zero at the stagnation point and approaches 1 asymptotically far downstream. The contribution of the vorticity term is given by a solution of Poisson's equation with source term equal to $\Omega sgn(y)$. A technique for solution of this problem is given in Ref. 1 and is applied in Ref. 2:

$$U_v = \frac{2\xi}{\pi}\frac{d\xi}{ds}\int_0^\infty \frac{(d\psi_b/d\xi^*)d\xi^*}{(\xi^*)^2 - \xi^2} + \Omega|y_b|\cos\theta_b(s) \tag{2}$$

The rotational contribution is given by $\Omega\,|y_b|\cos\theta_b$; θ_b is the angle the tangent to the body surface makes with the positive x axis. s is the dimensionless arc length measured from a stagnation point. The Cauchy principal integral represents the potential solution for the upper half plane. $\psi_b = (-\Omega|y_b|y_b/2)$ represents the boundary value for a Dirichlet problem. ξ is the real axis in the Z plane, and $Z = f(x + iy) = f(z)$ maps the body to the ξ axis. The complex potential $\bar{Z} = \bar{\xi} + i\bar{\eta} = g(z)$ for the uniform flow problem will map the body to a positive real axis. Then $Z^2 = \bar{Z}$ will complete the mapping of the body surface to the real axis. Advantage is taken of the fact that $d\psi_b/d\xi^*$ is even in ξ^*. The

Received February 16, 1965; revision received May 5, 1965. This research was supported by NASA Contract No. NsG 237-62.

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asymptotic form of U_v may be obtained by making the substitutions, $\chi^* = 1/\xi^*$, $\chi = 1/\xi$, and going to the limit $\chi \to 0$. The result is

$$U_v \sim (-2/\pi\xi)\,(d\xi/ds)\psi_b + \Omega|y_b| \cos\theta_b(s) \text{ as } \xi \to \infty \quad (3)$$

The limiting form of (3) may be completed with the aid of the following: $\cos\theta_b \sim 1$ as $\xi \to \infty$ follows from the equation of the body. $|d\bar{Z}/dz|_b = (d\xi^2/ds) = (d\bar{\xi}/ds) \sim 1$ as $\xi \to \infty$ follows from the fact that $\bar{Z}$ is the complex potential for uniform flow (dimensionless, $m < 1$) and from the asymptotic behavior of the contribution of the uniform flow to the velocity component tangent to the body surface. $\xi^2 \sim x_b \sim s$ as $\xi \to \infty$ follows from the asymptotic behavior of the complex potential $\bar{Z}$ and from the equation of the body. In view of the preceding, U_v may be written as

$$U_v \sim (2\Omega/\pi)\, x_b^{2m-1} + \Omega x_b^m \text{ as } x_b \to \infty \quad (4)$$

For $m < 1$, Ωx_b^m is the dominant term. Thus on bodies that grow slower than a wedge the first-order outer velocity component is

$$U_1 \sim \Omega\, s^m \qquad \text{as } s \to \infty \quad (5)$$

Far downstream external vorticity no longer contributes second-order terms[3,4] but becomes dominant for certain bodies. The problem of the flat plate is a special case of the problem considered here and was solved by Ting.[5] The method of inner and outer expansions[3,4] provides the differential equation and the boundary conditions far from the body:

$$\Psi_{1NNN} + \Psi_{1s}\Psi_{1NN} - \Psi_{1N}\Psi_{1Ns} = -\,U_1U_1' - \Omega R^{-1/2} d/ds\,(\delta_1 U_1) \quad (6)$$

Here R is an appropriate Reynolds number $R = U_0a/\nu$, ν is the kinematic viscosity, Ψ_1 is the first-order inner-stream function $[\Psi = R^{-1/2}\Psi_1 + 0(R^{-1})]$, $N = nR^{1/2}$ is the dimensionless normal distance to the wall, and δ_1 is a displacement thickness given by

$$\delta_1 = \int_0^\infty \left(\frac{1 + \Omega R^{-1/2} N}{U_1} - \frac{\Psi_{1N}}{U_1}\right) dN \quad (7)$$

Boundary conditions are given by

$$\Psi \sim [\Omega R^{-1/2}(N^2/2) + U_1 N - U_1\delta_1]R^{-1/2} + 0(R^{-1}) \sim \Psi_1 R^{-1/2} + 0(R^{-1}) \text{ as } N \to \infty \quad (8)$$

$$\Psi_1(0, s) = 0 \quad (9)$$

$$\Psi_{1N}(0, s) = 0 \quad (10)$$

The left-hand side of (6) is the usual first-order approximation to the Navier-Stokes equations. The pressure gradient is given by the right-hand side of (6). It is evaluated with the aid of the matching condition given to the first order by Eq. (8). For small s, $\Omega R^{-1/2}(N^2/2)$ is a second-order term. N effective for matching grows like $s^{1/2}$ (a boundary-layer thickness). Hence, for large s, $\Omega R^{-1/2}(N^2/2)$ will eventually become dominant as $s \to \infty$. This is remedied by incorporating $\Omega R^{-1/2}(N^2/2)$ as a first-order term in Eq. (8). Terms of order R^{-1}, which have been omitted, are for contributions caused by longitudinal curvature and the displacement effects.[3,4] Ψ would then be given by solutions for small s with external vorticity, considered as a second-order term.[2] For larger s a solution would be given by Eq. (6). The two cases could be matched for an appropriate value of s. The second-order problems of curvature and displacement could then be evaluated later. It is assumed that these second-order terms omitted will remain second-order terms as $s \to \infty$. For the curvature problem this is readily verified for $m < 1$ by using Eqs. (5, 12, and 14) and the omitted boundary term, which is $\varkappa U_1(N^2/2)R^{-1}$ ($\varkappa$ is the curvature). For the displacement problem verification of the assumption can be made for $m<1$ by a somewhat more involved argument.

A similarity form is given by

$$\Psi_1(N, s) = \Omega R^{-1/2}(N^2/2) + (\Omega/R^{1/2}\sigma^*)^{-1/3} f(\sigma^*, \eta) \quad (11)$$

where

$$\sigma^* = \int_0^s U_1^3\, ds \quad (12)$$

$$\sigma = \Omega\, R^{-1/2}(\sigma^*)^{1/2} \quad (13)$$

$$\eta = U_1 N(\Omega/R^{1/2}\sigma^*)^{1/3} \quad (14)$$

Substitution of (11–14) into (6) yields

$$f_{\eta\eta\eta} + \frac{1}{3}f + \frac{\sigma f_\sigma}{2} - \frac{1}{3}\eta f_\eta - \frac{1}{2}\eta\sigma f_{\sigma\eta} - \frac{\eta^2\sigma\,(dU_1/d\sigma)}{2\,U_1} f_{\eta\eta} + \frac{1}{3}\eta^2 f_{\eta\eta} + \frac{U_1^2}{\sigma^{2/3}}\left[\frac{1}{3}ff_{\eta\eta} + \frac{\sigma}{2}(f_\sigma f_{\eta\eta} - f_\eta f_{\eta\sigma} + \frac{1}{2}\left(\frac{\sigma dU_1/d\sigma}{U_1}\right) \times (1 - f_\eta^2)\right] = -\frac{1}{3}\Delta - \frac{1}{2}\sigma\frac{d\Delta}{d\sigma} \quad (15)$$

where

$$\Delta = \lim_{\eta\to\infty} (\eta - f) \quad (16)$$

Boundary conditions are

$$f(\sigma, 0) = 0 \quad (17a)$$

$$f_\eta(\sigma, 0) = 0 \quad (17b)$$

$$f_\eta(\sigma, \infty) = 1 \quad (17c)$$

Equation (15) may be used as the basis for the more general nonsimilar flow problem mentioned previously. The present note will be confined to the case when $\sigma \to \infty$. $U_1^2/\sigma^{2/3} \to 0$ if $U_1 \sim s^m$, where $m < \frac{1}{3}$. This is the case for similar flow as $s \to \infty$. If $m > \frac{1}{3}$, the external vorticity contributes only second-order terms, and ordinary boundary-layer theory is valid. This result could also have been obtained from Eq. (8) using $U_1 \sim s^m$ and $N \sim s^{1/2}$. The equation for similar flow is

$$f''' + \frac{1}{3}f - \frac{1}{3}\eta f' + \left[\frac{1}{3} - \frac{\sigma}{2}\left(\frac{dU_1/d\sigma}{U_1}\right)\right]\eta^2 f'' = -\frac{\Delta}{3} \quad (18)$$

Equation (18) may be solved approximately by recasting into an integral equation form and using a Kármán-Pohlhausen quartic polynomial approximation for f'. Integrating (18) from zero to ∞, one obtains

$$f''(0) = 2\left[\frac{2}{3} - \frac{\sigma}{2}\left(\frac{dU_1/d\sigma}{U_1}\right)\right]\int_0^\infty \eta(1 - f')d\eta \quad (19)$$

The quartic is given by

$$f' = 1 - (1 - \bar{\eta})^3(1 + \bar{\eta}) + (\lambda/6)\bar{\eta}(1 - \bar{\eta})^3 \quad \bar{\eta} \le 1$$
$$= 0 \quad \eta > 1 \quad (20)$$

where

$$\bar{\eta} = \eta/\bar{\delta} \quad (21)$$

$$\lambda = (\Delta/3)\bar{\delta}^2 \quad (22)$$

Substitution of (20) and (21) into (19) gives

$$\frac{1}{\bar{\delta}}\left(2 + \frac{\lambda}{6}\right) = 2\left[\frac{2}{3} - \frac{\sigma}{2}\left(\frac{dU_1/d\sigma}{U_1}\right)\right]\frac{\bar{\delta}^2}{15}\left(1 - \frac{\lambda}{24}\right) \quad (23)$$

Applying the limiting form of f at $\bar{\eta} = 1$, one obtains

$$\bar{\delta}\int_0^1 f'd\bar{\eta} = \bar{\delta} - \Delta = \bar{\delta} - \frac{3\lambda}{\bar{\delta}^2} = \bar{\delta}\left(\frac{7}{10} + \frac{\lambda}{120}\right) \quad (24)$$

In view of the requirement that $m < \frac{1}{3}$, the similar flow is defined for $0 < \sigma(dU_1/d\sigma)/U_1 < \frac{1}{3}$. A solution is obtained from Eqs. (22–24). The smallest value of $\bar{\delta}$ is chosen

$$\lambda = 12\{1 - [(4\text{-}6M)/(7\text{-}6M)]^{1/2}\}$$
$$\bar{\delta}^3 = 3\lambda/[\tfrac{3}{10} - (\lambda/120)] \quad (25)$$

$$f''(0) = (1/\bar{\delta})\,(2 + \lambda/6) \quad (26)$$

$$\Delta = \bar{\delta}(\tfrac{3}{10} - \lambda/20) \quad (27)$$

where

$$M = \sigma(d\,U_1/d\,\sigma)/U_1 \quad (28)$$

For $M = 0$, $\bar{\delta} = 3.17$, $f''(0) = 0.7846$, and $\Delta = 0.8739$. Ting gives $f''(0) = 0.7866$ and $\Delta = 0.8695$. For $M = \frac{1}{3}$ one obtains $f''(0) = 0.7411$ and $\Delta = 0.9715$. Comparison with Ting's solution for $M = 0$ shows surprisingly good accuracy for the approximate method. The variation of $f''(0)$ for $0 < M < \frac{1}{3}$ is quite small. The corresponding variation of Δ, and hence the profile (f', f'') is somewhat greater.

Thus Ting's solution is only one member of a similar family of flows. In fact, Ting's solution is also valid for bodies that attain a finite width as $s \to \infty$.

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Comments on "Stability of Damped Mechanical Systems" and a Further Extension

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PRINGLE'S note[1] yields an immediate proof of what I have elsewhere[2] called the Kelvin-Tait-Chetaev theorem. Also it generalizes the theorem in a different way from the generalization given in Ref. 2. Furthermore, for linear systems, Pringle's note suggests still another refinement. Since these results are very useful for satellite attitude dynamics, it may be helpful to compare and illustrate them by means of a simple example and to give a proof of the further refinement.

Consider the following system:

$$\begin{aligned} m_1\ddot{x}_1 + d_0\dot{x}_1 + d_1(\dot{x}_1 - \dot{x}_2) + g\dot{x}_2 + k_1x_1 &= 0 \\ m_2\ddot{x}_2 - d_1(\dot{x}_1 - \dot{x}_2) + d_0\dot{x}_2 - g\dot{x}_1 + k_2x_2 &= 0 \end{aligned} \quad (1)$$

The terms containing d_0 and d_1 are damping terms; those containing g arise from gyroscopic effects. System (1) is characteristic of larger systems that arise in small-angle attitude-control studies of the form:

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0 \quad (2)$$

where $\mathbf{A}$ is a symmetric, positive definite inertia matrix, $\mathbf{D}$ is a symmetric damping matrix, $\mathbf{G}$ is a skew-symmetric gyroscopic matrix, and $\mathbf{K}$ is a symmetric stiffness matrix. With little loss in generality, we shall assume throughout that $\mathbf{K}$ has no zero eigenvalues, so that in (1) $k_1k_2 \neq 0$.

Received April 22, 1965; revision received April 22, 1965.

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To investigate the stability of (2), one could calculate the roots s of the system's characteristic polynomial:

$$|s^2\mathbf{A} + s(\mathbf{D} + \mathbf{G}) + \mathbf{K}| = 0 \quad (3)$$

But if $\mathbf{D}$ is positive definite, the *KTC* (Kelvin-Tait-Chetaev) theorem gives the following simplification:

Theorem A: When $\mathbf{D}$ *is positive definite, Eq. (3) has all its roots in the open left half-plane, if* $\mathbf{K}$ *has all positive eigenvalues and at least one root in the open right half-plane, and if* $\mathbf{K}$ *has any negative eigenvalues.*

Thus, when $\mathbf{D}$ is positive definite, the character of the roots of (3) is determined completely by the character of the roots of $|\mathbf{K} - s\mathbf{I}| = 0$, where $\mathbf{I}$ is the unit matrix. Furthermore, in many cases the coordinates are coupled only via damping and gyroscopic terms, so that not only is $|\mathbf{K} - s\mathbf{I}| = 0$ of degree n instead of $2n$, but it is of much simpler structure than Eq. (3). For instance, in the example (1), $\mathbf{D}$ is clearly positive definite for $d_0 > 0$, $d_1 > 0$, and $\mathbf{K}$ has only the diagonal elements k_1, k_2. If k_1, k_2 are both positive, the KTC theorem says that all four roots lie in the open left half-plane; if k_1 or k_2 are negative, one or more roots is in the open right half-plane.

In Ref. 2, the KTC theorem is sharpened as follows:

Theorem B: When $\mathbf{D}$ *is positive definite, Eq. (3) has no roots on the imaginary axis and as many roots in the right half-plane as there are negative eigenvalues of* $\mathbf{K}$. Thus in the simple example if $d_0 > 0$, $d_1 > 0$, and if $k_1 < 0$, $k_2 < 0$, we can conclude that there are two roots in the right half-plane, two roots in the left half-plane, and no roots on the imaginary axis.

However, in attitude-control systems, Theorems A and B are often inapplicable. There are typically no "ground" dampers between the satellite parts and the reference frame; energy is dissipated only by relative motion between satellite parts. In this case, the damping matrix $\mathbf{D}$ in (2) is only positive semidefinite rather than positive definite, that is, the power dissipation function $P = -(\dot{\mathbf{x}}, \mathbf{D}\dot{\mathbf{x}})$ may be zero as well as negative for some choices of $\dot{\mathbf{x}}$. For example, if the "ground" dampers in (1) are absent, $d_0 = 0$, then $P = -d_1(\dot{x}_1 - \dot{x}_2)^2$, which is zero for $\dot{x}_1 = \dot{x}_2$.

As Pringle has pointed out in a private communication, the assumption of a positive definite $\mathbf{D}$ is too restrictive. In the design of attitude-control systems, one strives not for a positive definite damping matrix but rather for damping that affects the entire system, so that any motion induces energy dissipation. It is perhaps helpful to coin the term *pervasive* for this kind of damped system. More specifically, form the scalar product of Eq. (2) with $\dot{\mathbf{x}}$ to get

$$(d/dt)[\tfrac{1}{2}(\dot{\mathbf{x}}, \mathbf{A}\dot{\mathbf{x}} + \mathbf{x}, \mathbf{K}\mathbf{x})] = P = -(\dot{\mathbf{x}}, \mathbf{D}\dot{\mathbf{x}}) \quad (4)$$

Then we define a pervasive power dissipation P as follows: P is pervasive if it is nonpositive and can be zero for all $t > 0$ only when the system is at the equilibrium point for $t > 0$. In (1) damping is pervasive for $d_0 = 0$ and $g \neq 0$. To show this, we note that when $P = -d_1(\dot{x}_1 - \dot{x}_2)^2 \equiv 0$, we have $x_1 = x_2 + C$, where C is a constant. But Eqs. (1) then become

$$m_1\ddot{x}_1 + g\dot{x}_1 + k_1x_1 = 0 \qquad m_2\ddot{x}_1 - g\dot{x}_1 + k_2x_1 = C$$

which can be satisfied only when $x_1 = C = x_2 = 0$. On the other hand, if $g = 0$ and $k_1/m_1 = k_2/m_2$, the damping is not pervasive; m_1 and m_2 can remain at a fixed distance apart in an oscillation with zero power dissipation. Note that pervasiveness is a property of both the structure of P and the structure of the differential equations. In general, an argument such as the preceding is needed to show pervasiveness.

The generalization of the idea of pervasive damping is used by Pringle. The bracketed term in Eq. (4) is the Hamiltonian of system (2). In a general dynamical system, instead of (4) we have an equation of the form $dH/dt = P$, where both H, the Hamiltonian, and P are functions of the generalized coordinates and momenta q_i, p_i. We assume that the origin in the (q_i, p_i) space is an equilibrium point and adopt the pre-